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Real Interpolation and Measure of Weak Noncompactness

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Abstract. Behavior of weak measures of noncompactness under real interpolation is investigated. It is shown that "convexity type" theorems hold true for weak measures of noncompactness.

Preliminaries

1. Weak measure of noncompactness

Let X be an arbitrary Banach space and D be a bounded subset of X. The weak measure of noncompactness of D (the measure of noncompactness of D in the weak topology) $\omega(D)$, is defined as

$$\omega(D) = \omega_X(D) := \inf\{\varepsilon > 0 : D \subset \varepsilon U_X + W,\ W \subset X \text{ weakly compact}\}$$

where U_X denotes the closed unit ball of X. This concept was first introduced by De Blasi [9] and has been applied to obtain fixed point theorems [11] and existence results for differential and functional equations in Banach spaces (see [2], [3]).

Several facts will be useful in the following section:

1. $\omega(D) = 0$ if and only if D is relatively weakly compact.

2. $A \subset B$ implies $\omega(A) \le \omega(B)$.

3. $\omega(A + B) \le \omega(A) + \omega(B)$.

4. $\omega(\lambda A) = \lambda\omega(A)$ for $\lambda \ge 0$.

5. $\omega(A) \le \beta(A)$ where β denotes the usuall ball measure of noncompactness.

6. If $X_0 \subset X$ and $\|x\|_X \le c\,\|x\|_{X_0}$ for all $x \in X_0$, then $\omega_X(D) \le c\omega_{X_0}(D)$ for all $D \subset X_0$ bounded.

Also it is well known that $\omega(U_X) = 0$ if X is reflexive, $\omega(U_X) = 1$ if X is not reflexive [9].

Let $T \in L(X, Y)$ be a bounded linear map. T is called a weak k-set contraction ($k \geq 0$) if

$$\omega_Y(T(D)) \leq k\omega_X(D) \quad \text{for all bounded sets} \quad D \subset X .$$

Here ω_X, ω_Y are weak measures of noncompactness in X and Y, respectively.

The number

$$\omega(T) = \omega(T_{X,Y}) := \min\{k : T \text{ is a weak } k\text{-set contraction}\}$$

is called the measure of weak noncompactness of T. The $T_{X,Y}$ denotes the bounded linear map from X to Y.

Properties.

1. *If* $X_0 \subset X$ *and* $\|x\|_X \leq c\,\|x\|_{X_0}$ *for all* $x \in X_0$, *then* $\omega(T_{X_0,Y}) \leq c\omega(T_{X,Y})$.
2. *If* $Y_0 \subset Y$ *and* $\|y\|_Y \leq c\,\|y\|_{Y_0}$ *for all* $y \in Y_0$, *then* $\omega(T_{X,Y}) \leq c\omega T_{X,Y_0})$.
3. $\omega(T_{X,Y}) = \omega_Y(T(U_X)) := \inf\{r > 0 : T(U_X) \subset rU_Y + W_Y,\ W_Y \subset Y \text{ weakly compact}\}$.

Proof. 1. and 2. are staightforward. To prove 3., observe that

$$\omega(T_{X,Y}) = \sup\{\omega_Y(T(D)) : D \text{ is bounded, } \omega_X(D) = 1\} \geq \omega_Y(T(U_X)) ,$$

where $\sup \emptyset = 0$ by definition. Moreover, if D is bounded and $\omega_X(D) = 1$, then given any $\varepsilon > 0$ there exists a weakly compact set $W_X \subset X$ such that $D \subset (1 + \varepsilon)\,U_X + W_X$. Thus $T(D) \subset T((1 + \varepsilon)\,U_X + W_X)$. Using the subadditivity and the homogeneity property of ω together with the fact that $T(W_X)$ is a weakly compact set, we have

$$\omega(T(D)) \leq \omega[(1 + \varepsilon)\,T(U_X)] = (1 + \varepsilon)\,\omega(T(U_X)) .$$

Hence

$$\omega(T(D)) \leq (1 + \varepsilon)\,\omega_Y(T(U_X)) ,$$

and it is now immediate that $\omega(T_{X,Y}) \leq \omega_Y(T(U_X))$.

2. Real interpolation

Let $\{A_0, A_1\}$ be a pair of Banach spaces which are continuously embedded into some Hausdorf linear topological space X. Then the vector spaces $A_0 + A_1$ and $A_0 \cap A_1$ are Banach spaces with respect to the norms:

$$\|a\|_{A_0+A_1} = \inf\{\|a_0\| + \|a_1\| : a = a_0 + a_1,\ a_0 \in A_0,\ a_1 \in A_1\}$$

for $a \in A_0 + A_1$ and

$$\|a\|_{A_0 \cap A_1} = \max\{\|a\|_{A_0}, \|a\|_{A_1}\}$$

for $a \in A_0 \cap A_1$.

A Banach space A is called an intermediate space with respect to $\{A_0, A_1\}$ if

$$A_0 \cap A_1 \subset A \subset A_0 + A_1$$

and the corresponding embeddings are continous. If, in addition, every bounded operator in $A_0 + A_1$ that leaves A_0 and A_1 invariant also maps A boundedly into itself, then A is called an interpolation space for $\{A_0, A_1\}$. Let $L(\{A_0, A_1\}, \{B_0, B_1\})$ be the family of all linear maps $T: A_0 + A_1 \to B_0 + B_1$ such that the restriction of T to A_j is in $L(A_j, B_j)$ for $j = 0, 1$. If A and B are intermediate spaces with respect to $\{A_0, A_1\}$ and $\{B_0, B_1\}$ respectively, we say that A and B are interpolation spaces of exponent θ $(0 < \theta < 1)$ with respect to $\{A_0, A_1\}$ and $\{B_0, B_1\}$ if given any $T \in L(\{A_0, A_1\}, \{B_0, B_1\})$, the restriction of T to A is in $L(A, B)$ and

$$\|T_{A,B}\| \leq \|T_{A_0,B_0}\|^{1-\theta} \|T_{A_1,B_1}\|^{\theta}.$$

Several methods of constructing interpolation spaces of exponent θ with respect to Banach pairs $\{A_0, A_1\}$ and $\{B_0, B_1\}$ are known (see [22]). The real method is defined as follows $(0 < \theta < 1, 1 \leq p \leq \infty)$. If $p < \infty$,

$$A_{\theta,p} = (A_0, A_1)_{\theta,p} = \left\{a \in A_0 + A_1 : \|a\|_{\theta,p} = \left(\int_0^\infty [t^{-\theta}K(t, a)]^p \frac{dt}{t}\right)^{1/p} < \infty\right\}.$$

If $p = \infty$, then

$$A_{\theta,\infty} = \left\{a \in A_0 + A_1 : \|a\|_{\theta,p} = \sup_{0<t<\infty} t^{-\theta}K(t, a) < \infty\right\}$$

where $K(t, a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_0 \in A_0, a_1 \in A_1\}$.

It can be shown that if $a \in A_0 \cap A_1$, then

$$\|a\|_{\theta,p} \leq \|a\|_{A_0}^{1-\theta} \|a\|_{A_1}^{\theta}.$$

Finally, given A and B intermediate spaces with respect to $\{A_0, A_1\}$ and $\{B_0, B_1\}$, we say that A is of K-type θ, $\theta \in (0, 1)$, if there is a positive constant c such that for all $t > 0$,

$$K(t, a) \leq ct^{\theta} \|a\|_A \quad \text{for all} \quad a \in A.$$

B is of J-type θ if there is a positive constant C such that

$$\|b\|_B \leq C \|b\|_{B_0}^{\theta} \|b\|_{B_1}^{1-\theta} \quad \text{for all} \quad b \in B_0 \cap B_1.$$

Real interpolation $A_{\theta,p}$ spaces are of both K-type θ and J-type θ.

3. Some results on real interpolation of compact and weakly compact operators

In 1960, M. A. KRASNOSEL'SKII [13] proved the following version of the Riesz-Thorin theorem for compact operators: Let $T: L_{p_0} \to L_{q_0}$ be bounded and $T: L_{p_1} \to L_{q_1}$ be compact, where all four exponents are in the range $[1, \infty]$ and $q_0 < \infty$, then $T: L_{p_\theta} \to L_{q_\theta}$ is compact as well. Here, as usual, θ is any number in $(0, 1)$ and $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

In light of the theorem of KRASNOSEL'SKII, it is natural to ask, given two interpolation pairs $\{A_0, A_1\}$ and $\{B_0, B_1\}$ and interpolation spaces A and B obtained by the real method, with respect to $\{A_0, A_1\}$ and $\{B_0, B_1\}$, whether T, viewed as a map from A to B, inherits any compactness properties which it may possess as an element of $L(A_i, B_i)$.

An one sided answer to this question was given by LIONS and PEETRE [14]. They showed that if $B_0 = B_1$ and A is of K-type θ for some $\theta \in (0, 1)$, then $T: A \to B_0$ is compact if either $T: A_0 \to B_0$ or $T: A_1 \to B_0$ is compact. They also have a similar result when $B_0 \neq B_1$ and $A_0 = A_1$. The general case, in which $A_0 \neq A_1$ and $B_0 \neq B_1$, was solved by PERSSON [19]. He shows that if $T: A_0 \to B_0$ is compact then so is $T: A \to B$. To do this he was forced to make the assumption that $\{B_0, B_1\}$ have a certain approximation property. This restriction on $\{B_0, B_1\}$ was removed in the later work of HAYAKAWA [12], but at the expense of the additional hypothesis that $T: A_1 \to B_1$ is also compact. Subsequent work in this area deals with the problems of interpolating measures of noncompactness [10] or behavior of width numbers or entropy numbers under interpolation [20], [21]. Recently CWIKEL [7] showed that if $T: A_0 \to B_0$ is compact and $T: A_1 \to B_1$ is bounded, then $T: (A_0, A_1)_{\theta,p} \to (B_0, B_1)_{\theta,p}$ is compact too.

A bounded linear operator $T: A \to B$ between Banach spaces A and B is weakly compact if $T(U_A)$ is relatively weakly compact. Clearly if either A or B is reflexive, then every bounded linear operator $T: A \to B$ is weakly compact. Moreover, weakly compact operators factor through reflexive Banch spaces. This important results is due to DAVIS, FIGIEL, JOHNSON, and PELCZYNSKI [8] which states as:

If $T: A \to B$ is a weakly compact operator, then there is a reflexive Banach space C and bounded liner operators $S: A \to C$, $R: C \to B$ such that $RS = T$. B. BEAUZAMY in [4] has shown that for $0 < \theta < 1$ and $1 < p < \infty$ the real interpolation spaces $(A_0, A_1)_{\theta,p}$ are reflexive if and only if the imbedding $I: A_0 \cap A_1 \to A_0 + A_1$ is weakly compact. Similar results have been shown to be true for more general interpolation spaces [18].

Interpolation of weakly compact operators were also investigated by L. MALIGRANDA-A. QUEVEDO [16], YU. A. BRUDNYI-N. YA. KRUGLJAK [6], L. MALIGRANDA [15] and M. MASTYLO [17] where they show that given $1 < p < \infty$ and $T \in L(\{A_0, A_1\}, \{B_0, B_1\})$ one has that $T: (A_0, A_1)_{\theta,p} \to (B_0, B_1)_{\theta,p}$ is weakly compact if and only if $T: A_0 \cap A_1 \to B_0 + B_1$ is weakly compact. Their results generalizes the theorem of BEAUZAMY [4].

The results

Definition. A Banach couple $\{B_0, B_1\}$ with the property that $B_0 \cap B_1$ is dense in B_0 and B_1 us called *coujugate couple.*

It is known that if $\{B_0, B_1\}$ is a conjugate couple of Banach spaces, then $(B_0 \cap B_1)^*$ is isometrically isomorphic to $B_0^* + B_1^*$ [5].

Lemma. *Let $\{B_0, B_1\}$ be a Banach couple. Suppose that W_0 and W_1 are weakly compact sets contained in the spaces B_0 and B_1, respectively. Then $W_0 \cap W_1$ is weakly compact in $B_0 \cap B_1$.*

Proof. Let $C_i = (B_0 \cap B_1, \|\cdot\|_{B_i})$, $i = 0, 1$. Since $C_0 \cap C_1 = B_0 \cap B_1$ is dense in both C_0 and C_1, we have

$$(C_0 \cap C_1)^* = C_0^* + C_1^*$$

or equivalently $(B_0 \cap B_1, \|\cdot\|_{B_0 \cap B_1})^* = (B_0 \cap B_1, \|\cdot\|_{B_0})^* + (B_0 \cap B_1, \|\cdot\|_{B_1})^*$. Take a sequence of elements (x_n) in $W_0 \cap W_1$. Since W_0 is weakly compact, there is a subsequence

$(x_{n,k})$ of (x_n) such that $x_{n,k} \to x$ (weakly) in B_0. But $(x_{n,k})$ is also weakly compact in $W_1 \subset B_1$, so there exists a subsequence $(x_{n,k,j})$ of $(x_{n,k})$ such that $x_{n,k,j} \to x$ (weakly) in B_1. It follows that $x_{n,k,j} \to x$ (weakly) in C_0 and C_1 ($W_0 \cap W_1 \subset C_0 \cap C_1 = B_0 \cap B_1$). Next we observe that $x_{n,k,j} \to x$ (weakly) in $C_0 \cap C_1$, because given $x^* \in (C_0 \cap C_1)^*$, say $x^* = x_0^* + x_1^*$ with $x_0^* \in C_0^*$, $x_1^* \in C_1^*$ one gets

$$\langle x^*, x_{n,k,j}\rangle = \langle x_0^*, x_{n,k,j}\rangle + \langle x_1^*, x_{n,k,j}\rangle \to \langle x^*, x\rangle .$$

Since $C_0 \cap C_1 = B_0 \cap B_1$, we have that $x_{n,k,j} \to x$ (weakly) in $B_0 \cap B_1$ and so $W_0 \cap W_1$ is weakly compact in $B_0 \cap B_1$. □

Using the above Lemma, we obtain the following theorem, which can be considered as a Riesz-Thorin type theorem for measure of weak noncompactness. Similar results for ordinary measure of noncompactness can be found in [10].

Theorem 1. *Let $\{B_0, B_1\}$ be a Banach couple and A be a Banach space. Suppose that B is of J-type θ for some $\theta \in (0, 1)$. If $T \in L(\{A, A\}, \{B_0, B_1\})$, then*

$$\omega(T_{A,B}) \leq C(\omega(T_{A,B_0}))^{1-\theta}\,(\omega(T_{A,B_1}))^{\theta} .$$

Proof. Let D be a bounded subset of A. Setting $k_0 = \omega(T_{A,B_0})$ and $k_1 = \omega(T_{A,B_1})$ we have $\omega_{B_i}(T(D)) \leq k_i\delta$ where $\delta = \omega_A(D)$ and $i = 0, 1$. From the definition of weak measure of noncompactness we have that $T(D) \subset (k_0\delta)\,U_{B_0} + W_0$, W_0 is a weakly compact set in B_0, and similarly, $T(D) \subset (k_1\delta)\,U_{B_1} + W_1$, W_1 is a weakly compact set in B_1. In other words we have $\|Tx - w_0\|_{B_0} \leq k_0\delta$ and $\|Tx - w_1\|_{B_1} \leq k_1\delta$ where $w_0 \in W_0$, $w_1 \in W_1$. Since B is of J-type θ,

$$\|Tx - w'\|_B \leq C\,\|Tx - w'\|_{B_0}^{1-\theta}\,\|Tx - w'\|_{B_1}^{\theta} \quad \text{for all} \quad w' \in W_0 \cap W_1$$

i.e.,

$$T(D) \subset k_0^{1-\theta}k_1^{\theta}\delta(U_{B_0} \cap U_{B_1}) + W_0 \cap W_1 .$$

By the above lemma $W_0 \cap W_1$ is weakly compact in $B_0 \cap B_1$. Then $W_0 \cap W_1$ is also weakly compact in B since $B_0 \cap B_1 \subset B$, and the result follows. □

Remark. The above theorem does not use the linearity of T, therefore under suitable hypothesis, one can obtain a similar result for nonlinear weak k-set contraction.

Theorem 2. *Let $\{A_0, A_1\}$ be a Banach couple and B be a Banach space. Suppose that A is of K-type θ for some $\theta \in (0, 1)$. If $T \in L(\{A_0, A_1\}, \{B, B\})$, then*

a. $$\omega(T_{A,B}) \leq c(1-\theta)^{(\theta-1)}\,\theta^{-\theta}\omega(T_{A_0,B})\,\omega(T_{A_1,B})^{\theta} ,$$

b. $$\omega(T_{A,B}) \leq \omega(T_{A_0 \cap A_1,B}) \leq \frac{\left(\frac{1-\theta}{\theta}\right)^{\theta} + \left(\frac{\theta}{1-\theta}\right)^{1-\theta}}{4c}\,\omega(T_{A_0 \cap A_1,B})^{1-\theta}\,d^{\theta}$$

where $d = \max\,(\|T_{A_0,B}\|, \|T_{A_1,B}\|)$.

Proof. a. Let D be a bounded subset of A and let $t > 0$. Since A is of K-type θ, given any $a \in D$ there exists $a_0 \in A_0$, $a_1 \in A_1$, such that $a = a_0 + a_1$, and

$$\|a_0\|_{A_0} \leq ct^{\theta}\,\|a\|_A , \qquad \|a_1\|_{A_1} \leq ct^{\theta-1}\,\|a\|_A \quad \text{for} \quad i = 0, 1 .$$

Let $D_i = \{a \in A_i : \|\alpha_i\|_{A_i} \leq ct^{\theta - i} \|a\|_A\}$ for $i = 0, 1$. Observe that the D_i's are nonempty and $D \subset D_0 + D_1$. Using the inequalities above we obtain

$$\omega_B(T(D_0)) \leq k_0 \omega_{A_0}(D_0) \leq k_0 ct^\theta \omega_A(D),$$
$$\omega_B(T(D_1)) \leq k_1 \omega_{A_1}(D_1) \leq k_1 ct^{\theta-1} \omega_A(D),$$

where $k_0 = \omega(T_{A_0, B})$ and $k_1 = \omega_B(T_{A_1, B})$. Since $D \subset D_0 + D_1$ and T is linear, it follows that

$$\begin{aligned}\omega_B(T(D)) \leq \omega_B(T(D_0 + D_1)) &\leq \omega_B(T(D_0)) + \omega_B(T(D_1)) \\ &\leq c(k_0 t^\theta + k_1 t^{\theta-1})\, \omega_A(D).\end{aligned}$$

Hence $\omega(T_{A,B}) \leq c(k_0 t^\theta + k_1 t^{\theta-1})$ for all $t > 0$. Minimizing the right-hand side over $t > 0$ we obtain the result.

b. Define, as usual, the spaces $\Delta(\tilde{A}) = A_0 \cap A_1$ and $\Sigma(\tilde{A}) = A_0 + A_1$ with the norms defined as in Section 2 above. Since A is said to be of K-type θ, $\theta \in (0, 1)$, there is a positive constant c such that for all $a \in A$ and all $t > 0$,

$$K(t, a) \leq ct^\theta \|a\|_A.$$

Fix $\varepsilon > 0$. Then there exists $t > 1$ such that $t^{-\theta} < (\varepsilon/4)\, c^{-1}$ and $t^{\theta-1} < (\varepsilon/4)\, c^{-1}$. If $x \in U_A$, then from above we obtain

$$\|a_0\|_{A_0} + t^{-1} \|a_1\|_{A_1} \leq t^{-\theta} + \varepsilon/4 \leq \varepsilon/2,$$
$$\|a'_0\|_{A_0} + t\, \|a'_1\|_{A_1} \leq ct^\theta + \varepsilon/4 \leq t\varepsilon/2,$$

where $a = a_0 + a_1 = a'_0 + a'_1$ is such that $a_0, a'_0 \in A_0$, $a_1, a'_1 \in A_1$.

Observe that $\|a_0\|_{A_0} < \varepsilon/2$, $\|a_1\|_{A_1} < t\varepsilon/2$, $\|a'_0\|_{A_0} < t\varepsilon/2$, and $\|a'_1\|_{A_1} < \varepsilon/2$. Let $b = a - a_0 - a'_1$. Then $b = a'_0 - a_0 \in A$ and $\|b\|_{A_0} \leq \|a_0\|_{A_0} + \|a'_0\|_{A_0} < t\varepsilon$. But $b = a_1 - a'_1 \in A_1$ and $\|b\|_{A_1} \leq \|a_1\|_{A_1} + \|a'_1\|_{A_1} < t\varepsilon$. Therefore $b \in A_0 \cap A_1$ and $\|b\|_{\Delta(\tilde{A})} \leq t\varepsilon$. Now since $a - b = a_0 + a'_1$ and $\|a - b\|_{\Sigma(\tilde{A})} \leq \|a_0\|_{A_0} + \|a'_1\|_{A_1} < \varepsilon$, we have $a = b + a_0 + a'_1$ with $b \in \Delta(\tilde{A})$, and $\|b\|_{\Delta(\tilde{A})} \leq t\varepsilon$, $\|a_0 + a'_1\|_{\Sigma(\tilde{A})} < \varepsilon$. Thus for every $\varepsilon > 0$ there is $t > 1$ such that

$$(*) \qquad U_A \subset \frac{1}{t\varepsilon}\, U_{\Delta(\tilde{A})} + \varepsilon^{-1} U_{\Sigma(\tilde{A})}.$$

On the other hand $T \in L(\{A_0, A_1\}, \{B, B\})$. Therefore $T(U_{\Sigma(\tilde{A})}) \subset dU_B$, where $d = \max(\|T_{A_0, B}\|, \|T_{A_1, B}\|)$. Suppose $r > \omega(T_{\Delta(\tilde{A}), B})$. Then $T(U_{\Delta(\tilde{A})}) \subset rU_B + W_B$ with W_B a weakly compact set in B. Using equation $(*)$ above, we have that

$$T(U_A) \subset \frac{1}{t\varepsilon}\, T(U_{\Delta(\tilde{A})}) + \varepsilon^{-1} T(U_{\Delta(\tilde{A})})$$

and

$$T(U_A) \subset \frac{1}{t\varepsilon}\,(rU_B + W_B) + \varepsilon^{-1} dU_B = \left(\frac{r}{t\varepsilon} + \varepsilon^{-1} d\right) U_B + \frac{1}{t\varepsilon}\, W_B,$$

which implies that

$$\omega(T_{A,B}) \leq \frac{r}{t\varepsilon} + \varepsilon^{-1} d.$$

Since $t^{\theta-1} < \varepsilon/4c$, we have

$$\omega(T_{A,B}) \leq (r/t + d)\, t^{1-\theta}/4c = (rt^{-\theta} + dt^{1-\theta})/4c\,,$$

therefore

$$\omega(T_{A_0 \cap A_1, B}) \leq [\omega(T_{A_0 \cap A_1, B})\, t^{-\theta} + dt^{1-\theta}]/4c\,.$$

Minimizing the right-hand side over $t > 1$, we obtain

$$\omega(T_{A_0 \cap A_1, B}) \leq \frac{\left(\dfrac{1-\theta}{\theta}\right)^{\theta} + \left(\dfrac{\theta}{1-\theta}\right)^{1-\theta}}{4c}\, \omega(T_{A_0 \cap A_1, B})^{1-\theta} d^{\theta}\,. \quad \square$$

Let $W(A, B)$ denote the space of weakly compact operators from A to B. The corresponding quotient norm is $\|T\|_\omega = \operatorname{dist}(T, W(A, B))$. Recall that ω and $\|\cdot\|_\omega$ are submultiplicative seminorms on $L(A, B)$ and that $\omega(T) \leq \|T\|_\omega$ for any operator T. In [1] it is shown that the seminorms ω and $\|T\|_\omega$ are equivalent in $L(A, B)$ if and only if B has the following weak compact approximation property (W.A.P. for short).

We say that a Banach space B has the W.A.P. if there is a $\lambda \geq 1$ such that for any weakly compact set $S \subset B$ and any $\varepsilon > 0$ there is a weakly compact operator $R: B \to B$ with

$$\sup_{x \in S} \|x - Rx\| \leq \varepsilon \quad \text{and} \quad \|id - R\| \leq \lambda\,.$$

Corollary. a. *Let $\{B_0, B_1\}$ be a Banach couple and A be a Banach space. Suppose that B is of J-type θ for some $\theta \in (0, 1)$ and has W.A.P. If $T \in L(\{A, A\}, \{B_0, B_1\})$, then*

$$\|T_{A,B}\|_\omega \leq C_1\, \|T_{A,B_0}\|_\omega^{1-\theta}\, \|T_{A,B_1}\|_\omega^{\theta}\,.$$

b. *Let $\{A_0, A_1\}$ be a Banach couple and B be a Banach space with W.A.P.. Suppose that A is of K-type θ for some $\theta \in (0, 1)$. If $T \in L(\{A_0, A_1\}, \{B, B\})$, then*

$$\|T_{A,B}\|_\omega \leq c_1 (1-\theta)^{\theta-1}\, \theta^{-\theta}\, \|T_{A_0,B}\|_\omega^{1-\theta}\, \|T_{A_1,B}\|_\omega^{\theta}\,.$$

Proof. Using the fact that the space B has W.A.P., we have

$$\omega(T_{A,B}) \leq \|T_{A,B}\|_\omega \leq \lambda\omega(T_{A,B})\,.$$

Now apply Theorem 1 and Theorem 2a to obtain the above inequalities. $\square$

Remark. Astala and Tylli [1] proved that if B is an $\mathscr{L}^p$ space for $p = 1$ or $p = \infty$ (for notation see [1]), then B has the weak approximation property if and only if B has the Schur property. It is well known that among the examples of spaces mentioned above ℓ^1 has the Schur property.

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